Pearson Edexcel

Mark Scheme (Results)

January 2024

Pearson Edexcel International Advanced Level
In Further Pure Mathematics F3 (WFM03)
Paper 01

Edexcel and BTEC Qualifications

Pearson: helping people progress, everywhere

January 2024
Question Paper Log Number P73488A
Publications Code WFM03_01_2401_MS

General Marking Guidance

- All candidates must receive the same treatment. Examiners must mark the first candidate in exactly the same way as they mark the last.
- Mark schemes should be applied positively. Candidates must be rewarded for what they have shown they can do rather than penalised for omissions.
- Examiners should mark according to the mark scheme not according to their perception of where the grade boundaries may lie.
- There is no ceiling on achievement. All marks on the mark scheme should be used appropriately.
- All the marks on the mark scheme are designed to be awarded. Examiners should always award full marks if deserved, i.e. if the answer matches the mark scheme. Examiners should also be prepared to award zero marks if the candidate's response is not worthy of credit according to the mark scheme.
- Where some judgement is required, mark schemes will provide the principles by which marks will be awarded and exemplification may be limited.
- When examiners are in doubt regarding the application of the mark scheme to a candidate's response, the team leader must be consulted.
- Crossed out work should be marked UNLESS the candidate has replaced it with an alternative response.

PEARSON EDEXCEL IAL MATHEMATICS

General Instructions for Marking

1. The total number of marks for the paper is 75

2. The Edexcel Mathematics mark schemes use the following types of marks:

- **M** marks: Method marks are awarded for 'knowing a method and completing an attempt to apply it', unless otherwise indicated.
- **A** marks: Accuracy marks can only be awarded if the relevant method (M) marks have been earned.
- **B** marks are unconditional accuracy marks (independent of M marks)
- Marks should not be subdivided.

3. Abbreviations

   These are some of the traditional marking abbreviations that will appear in the mark schemes.

   - ft – follow through
   - cao – correct answer only
   - cso - correct solution only. There must be no clear errors in this part of the question to obtain this mark
   - isw – ignore subsequent working
   - awrt – answers which round to
   - SC: special case
   - oe – or equivalent
   - dM – dependent method mark
   - dp decimal places
   - sf significant figures
   - ∗ The answer is given on the paper – apply cso

4. All A marks are 'correct answer only' (cao), unless shown, for example, as A1 ft to indicate that previous wrong working is to be followed through. After a misread however, the subsequent A marks affected are treated as A ft, but manifestly absurd answers should never be awarded A marks.

5. For misreading which does not alter the character of a question or materially simplify it, deduct two from any A or B marks gained, in that part of the question affected.

6. If a candidate makes more than one attempt at any question:
   - If all but one attempt is crossed out, mark the attempt which is NOT crossed out provided it is not cursory.
   - If either all attempts are crossed out or none are crossed out, mark all the attempts and score the highest single attempt.

7. Ignore wrong working or incorrect statements following a correct answer unless the mark scheme indicates otherwise.

<u>Usual rules for the method mark for solving a 3 term quadratic:</u>

(Note: There may be schemes where the below does not apply)

If no method is shown then one root must be obtained that is consistent with their equation.

1. Factorisation

$(x^2 + bx + c) = (x + p)(x + q),\ \text{where} |pq| = |c|$, leading to x = ...

$(ax^2 + bx + c) = (mx + p)(nx + q),\ \text{where} |pq| = |c| \text{ and } |mn| = |a|$, leading to x = ...

2. Formula

Complete attempt to use the correct formula with values for a, b and c leading to x = ... (may be unsimplified). Only allow slips if correct formula quoted first.

3. Completing the square (where a = 1; see scheme if a ≠ 1)

Solving $x^2 + bx + c = 0$: $\left(x \pm \frac{b}{2}\right)^2 \pm q \pm c = 0,\ \ q \neq 0$, leading to x = ...

<table>
<tr><th>Question Number</th><th>Scheme</th><th>Notes</th><th>Marks</th></tr>
<tr><td rowspan="4">1(i)</td><td>$(8)\int \frac{1}{16+x^2}\,dx = (8)\left(\frac{1}{4}\arctan\left(\frac{x}{4}\right)\right)$</td><td>Obtains …arctan (kx)
Allow k = 1</td><td>M1</td></tr>
<tr><td>$2\left[\arctan\left(\frac{x}{4}\right)\right]_4^{4\sqrt{3}} = 2(\arctan\sqrt{3} - \arctan 1) = ...$</td><td>Substitutes the given limits, subtracts either way round and obtains a value (could be a decimal). The substitution does not need to be seen explicitly and may be implied by their value.</td><td>dM1</td></tr>
<tr><td colspan="2">$\frac{\pi}{6}$ or $p = \frac{1}{6}$ Correct exact value (or value for p)
Accept equivalent exact expressions e.g. $\frac{2\pi}{12}$ or $p = \frac{2}{12}$ and isw if necessary.</td><td>A1</td></tr>
<tr><td colspan="2"></td><td>(3)</td></tr>
<tr><td rowspan="4">(ii)</td><td colspan="2">$2\int \frac{1}{\sqrt{9-4x^2}}\,dx = 2\left(\frac{1}{2}\arcsin\frac{2x}{3}\right)\left(\text{or e.g. } \arcsin\frac{x}{3/2}\right)$
M1: Obtains ...arcsin (kx). Allow k = 1 so allow just arcsin x.
A1: Fully correct integration but allow unsimplified as above</td><td>M1 A1</td></tr>
<tr><td colspan="2">$\left[\arcsin\left(\frac{2x}{3}\right)\right]_{\frac{3}{4}}^{k} = \arcsin\left(\frac{2k}{3}\right) - \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{12}$
$\Rightarrow \arcsin\left(\frac{2k}{3}\right) = \frac{\pi}{12} + \frac{\pi}{6} \Rightarrow \frac{2k}{3} = \sin\left(\frac{\pi}{4}\right) \Rightarrow \frac{2k}{3} = \frac{\sqrt{2}}{2} \Rightarrow k = ...$
Substitutes the given limits, subtracts either way round, sets = $\frac{\pi}{12}$, uses $\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$ and the correct order of operations condoning sign errors only to reach a value for k e.g.
$\pm\alpha\left(\arcsin\left(\frac{2k}{3}\right) - \frac{\pi}{6}\right) = \frac{\pi}{12} \Rightarrow \arcsin\left(\frac{2k}{3}\right) = \frac{\pi}{12\alpha} \pm \frac{\pi}{6} \Rightarrow k = \frac{3\sin\left(\frac{\pi}{12\alpha} \pm \frac{\pi}{6}\right)}{2}$
Note that k may be inexact (decimal) or may be in terms of “sin” but must have a simplified argument e.g. $k = \frac{3\sin\left(\frac{\pi}{4}\right)}{2}$</td><td>dM1</td></tr>
<tr><td colspan="2">$k = \frac{3\sqrt{2}}{4}$ or exact equivalent e.g., $\frac{3}{2\sqrt{2}}$
Note that a common incorrect answer is $k = \frac{3}{2}\sin\left(\frac{5\pi}{24}\right)(= 0.913...)$ which comes from an incorrect integral of $2\arcsin\left(\frac{2x}{3}\right)$ (generally scoring 1010)
Condone $x = \frac{3\sqrt{2}}{4}$</td><td>A1</td></tr>
<tr><td colspan="2"></td><td>(4)</td></tr>
<tr><td></td><td colspan="2"></td><td>Total 7</td></tr>
</table>

Question Number	Scheme	Notes	Marks
2(a) **Way 1** **TU = I**	$\mathbf{TU}=\mathbf{I}\Rightarrow\begin{pmatrix}2&3&7\\3&2&6\\a&4&b\end{pmatrix}\begin{pmatrix}6&-1&-4\\15&c&-9\\-8&a&5\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$ $\Rightarrow$ e.g., $\begin{matrix}6a+60-8b=0\\-4a-36+5b=1\end{matrix}$ or $\begin{matrix}-2+3c+7a=0\\-3+2c+6a=1\end{matrix}$ Obtains at least 2 equations with at least one correct. (condone column × row multiplication leading to the way 2 equations – see below). Ignore errors in unused elements or equations.		M1
	e.g., $\begin{matrix}6a-8b=-60\\-4a+5b=37\end{matrix}\Rightarrow a=\ldots,\ b=\ldots$ **or** $\begin{matrix}7a+3c=2\\6a+2c=4\end{matrix}\Rightarrow a=\ldots,\ c=\ldots$ Obtains values for two of a, b and c. You do **not** need to check their values. As long as the previous M mark was scored, it is sufficient to just write down values.		**d**M1
	$a=2,\ b=9,\ c=-4$	A1: Two correct values A1: All three correct values and no extra values unless they are rejected.	A1 A1
			(4)
Way 2 **UT = I** **For first 2 marks**	$\mathbf{UT}=\mathbf{I}\Rightarrow\begin{pmatrix}6&-1&-4\\15&c&-9\\-8&a&5\end{pmatrix}\begin{pmatrix}2&3&7\\3&2&6\\a&4&b\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$ $12-3-4a=1$ $\Rightarrow$ e.g., $42-6-4b=0$ $[45+2c-36=1]$ Obtains at least 2 equations with at least one correct. (condone column × row multiplication leading to the way 1 equations – see above). Ignore errors in unused elements or equations.		M1
	e.g., $-4a=-8$, $-4b=-36$ $[2c=-8]$ $\Rightarrow a=\ldots,\ b=\ldots$ Obtains values for two of a, b and c. You do **not** need to check their values. As long as the previous M mark was scored, it is sufficient to just write down values.		**d**M1

Way 3 **Inverses** **For first mark**	$\mathbf{T}^{-1}=\mathbf{U}\Rightarrow\frac{1}{4a-5b+36}\begin{pmatrix}2b-24 & -3b+28 & 4\\ 6a-3b & -7a+2b & 9\\ -2a+12 & 3a-8 & -5\end{pmatrix}=\begin{pmatrix}6 & -1 & -4\\ 15 & c & -9\\ -8 & a & 5\end{pmatrix}$ $\Rightarrow$ e.g., $\frac{4}{4a-5b+36}=-4,\ \frac{2b-24}{4a-5b+36}=6\ \left[\frac{-7a+2b}{4a-5b+36}=c\right]$ For $\mathbf{T}^{-1}=\frac{1}{\mathrm{f}(a,b)}\mathbf{M}$ where **M** has at least 1 correct element **and** obtains 2 equations. Note that there is no requirement to find all the elements of **M**. **OR** $\mathbf{U}^{-1}=\mathbf{T}\Rightarrow\frac{1}{-6a-2c+3}\begin{pmatrix}9a+5c & -4a+5 & 4c+9\\ -3 & -2 & -6\\ 15a+8c & -6a+8 & 6c+15\end{pmatrix}=\begin{pmatrix}2 & 3 & 7\\ 3 & 2 & 6\\ a & 4 & b\end{pmatrix}$ $\Rightarrow$ e.g., $\frac{-3}{-6a-2c+3}=3,\ \frac{4c+9}{-6a-2c+3}=7\ \left[\frac{6c+15}{-6a-2c+3}=b\right]$ For $\mathbf{U}^{-1}=\frac{1}{\mathrm{f}(a,c)}\mathbf{M}$ where **M** has at least 1 correct element **and** obtains 2 equations Note that there is no requirement to find all the elements of **M**.	M1

2(b)	$\frac{x-1}{3}=\frac{y}{-4}=z+2 \Rightarrow [l_2 : \mathbf{r} =]\begin{pmatrix}1\\0\\-2\end{pmatrix}\pm\lambda\begin{pmatrix}3\\-4\\1\end{pmatrix}\left(\text{or } \left(\mathbf{r}-\begin{pmatrix}1\\0\\-2\end{pmatrix}\right)\times\begin{pmatrix}3\\-4\\1\end{pmatrix}\right)=\mathbf{0}$ Obtains parametric/vector form (allow one slip only) or clearly identifies position and direction vectors. May be implied by an attempt to transform both.		M1
	$\begin{pmatrix}6&-1&-4\\15&'-4'&-9\\-8&'2'&5\end{pmatrix}\begin{pmatrix}1+3\lambda\\-4\lambda\\-2+\lambda\end{pmatrix}=\begin{pmatrix}6+18\lambda+4\lambda+8-4\lambda\\15+45\lambda+16\lambda+18-9\lambda\\-8-24\lambda-8\lambda-10+5\lambda\end{pmatrix}$ **or** their **U** × their $\begin{pmatrix}1&3\\0&-4\\-2&1\end{pmatrix}$ **or** ×their $\begin{pmatrix}1\\0\\-2\end{pmatrix}$ **and** ×their $\begin{pmatrix}3\\-4\\1\end{pmatrix}$ **or** their **U** ×their $\begin{pmatrix}1\\0\\-2\end{pmatrix}$ **and U** ×e.g. $\begin{pmatrix}4\\-4\\-1\end{pmatrix}$ then $dir=\begin{pmatrix}32\\85\\-45\end{pmatrix}-\begin{pmatrix}14\\33\\-18\end{pmatrix}$ Complete and correct method with their *b* and *c* for their **U** × their parametric form or **U** × both vectors or **U** × 2 points on the line and attempts direction. Must be an attempt to mutliply correctly i.e. clearly not row×row but allow attempts that use $\mathbf{T}^{-1}$ for **U** using their *a* and *b* provided all elements are constants and it is a "changed" **T** **OR** $\begin{pmatrix}2&3&7\\3&2&6\\"2"&4&"9"\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1+3\lambda\\-4\lambda\\-2+\lambda\end{pmatrix}\Rightarrow \begin{matrix}2x+3y+7z=1+3\lambda\\3x+2y+6z=-4\lambda\\2x+4y+9z=-2+\lambda\end{matrix}$ $\Rightarrow \begin{matrix}x=18\lambda+14\\y=52\lambda+33\\z=-18-27\lambda\end{matrix}$ A complete method using their parametric form and their **T** to produce and solve 3 simultaneous equations to find *x*, *y* and *z* in terms of λ Alternatively solves $\mathbf{T}\boldsymbol{x}=("\mathbf{i}-2\mathbf{k}")$ and $\mathbf{T}\boldsymbol{x}=("3\mathbf{i}-4\mathbf{k}+\mathbf{k}")$ to find position and direction		M1
	$[l_1 : \mathbf{r} =]\begin{pmatrix}14+18\lambda\\33+52\lambda\\-18-27\lambda\end{pmatrix}$ $\Rightarrow \frac{x-14}{18}=\frac{y-33}{52}=\frac{z+18}{-27}$	**d**M1: Correctly converts their result into Cartesian equation. **Requires previous method mark** A1: Correct Cartesian equation - allow equivalents e.g., $...=\frac{z-(-18)}{-27},\quad ...=\frac{-z-18}{27}$	**d**M1 A1
			(4)
			Total 8

2(b) Alternative

$$x = t \Rightarrow y = \frac{4}{3} - \frac{4}{3}t,\ \ z = \frac{1}{3}t - \frac{7}{3}$$

M1: Obtains parametric form (allow one slip only)

$$\begin{pmatrix} 6 & -1 & -4 \\ 15 & '-4' & -9 \\ -8 & '2' & 5 \end{pmatrix}\begin{pmatrix} t \\ \frac{4}{3} - \frac{4}{3}t \\ \frac{1}{3}t - \frac{7}{3} \end{pmatrix} = \begin{pmatrix} 6t - \frac{4}{3} + \frac{4}{3}t - \frac{4}{3}t + \frac{28}{3} \\ 15t - \frac{16}{3} + \frac{16}{3}t - 3t + 21 \\ -8t + \frac{8}{3} - \frac{8}{3}t + \frac{5}{3}t - \frac{35}{5} \end{pmatrix}$$

M1: As above

$$[l_1 : \mathbf{r} =]\begin{pmatrix} 8 + 6t \\ \frac{47}{3} + \frac{52}{3}t \\ -9 - 9t \end{pmatrix}$$

$$\Rightarrow \frac{x - 8}{6} = \frac{y - \frac{47}{3}}{\frac{52}{3}} = \frac{z + 9}{-9}$$

dM1A1: As above

Question Number	Scheme	Notes	Marks
3(a)(i)	$(\pm 7e, 0)$	Correct **coordinates** or $x = \pm 7e$, $y = 0$	B1
(ii)	$x = \pm\frac{7}{e}$	Correct **equations**	B1
	SC: If "49" used for "7" consistently in (i) and (ii) score B0 B1		
			(2)
(b)(i)	$\left(PS^2 =\right)(x - '7e')^2 + y^2$ oe e.g. $\left(PS^2 =\right)('7e' - x)^2 + y^2$	Correct expression or equivalent with their $7e$. Must be in terms of e, x and y only. Apply isw once a correct expression is seen.	B1ft
(ii)	$\left(PM^2 =\right)\left('\frac{7}{e}' - x\right)^2$ oe e.g. $\left(x - '\frac{7}{e}'\right)^2$	Correct expression or equivalent with their $\frac{7}{e}$. Must be in terms of e and x only. Apply isw once a correct expression is seen.	B1ft
			(2)
(c)	$\left(\frac{PS}{PM} = e \Rightarrow\right) PS^2 = e^2 PM^2 \Rightarrow (x - '7e')^2 + y^2 = e^2\left('\frac{7}{e}' - x\right)^2$ $\Rightarrow x^2 - 14ex + 49e^2 + y^2 = 49 - 14ex + e^2x^2$ Applies $PS^2 = e^2 PM^2$ with their PS and PM and expands (condone poor squaring)		M1
	$x^2(1-e^2) + y^2 = 49(1-e^2)$ $\Rightarrow \frac{x^2}{49} + \frac{y^2}{49(1-e^2)} = 1 \Rightarrow b^2 = 49(1-e^2)$ *	Reaches given answer with fully correct proof. All shown steps required. Note that it is possible to obtain this result even if the B marks are not scored in (b) e.g. correct expressions but not in the forms required.	A1*
			(2)
(d)	$(4\sqrt{3})^2 = 49(1-e^2) \Rightarrow e^2 \ldots$ or $e = \ldots$	Replaces b^2 with $(4\sqrt{3})^2$ and solves for e^2 or e.	M1
	$e = \frac{1}{7}$	Correct exact value for e (Not $\pm$)	A1
			(2)

(e)	$x=\dfrac{7}{2}\Rightarrow\dfrac{\left(\frac{7}{2}\right)^2}{49}+\dfrac{y^2}{'48'}=1\Rightarrow y=...\left[(\pm)6\right]$	Substitutes into their ellipse equation and obtains a value for y	M1
	Area $\Delta OPM=\left(\dfrac{1}{2}\right)\left(\dfrac{7}{'\left(\frac{1}{7}\right)'}-\dfrac{7}{2}\right)('6')=...$ Correct method for area of triangle OPM with their $\dfrac{7}{e}$ and their 6 May see other approaches, e.g., "shoelace" method e.g. $\dfrac{1}{2}\begin{vmatrix}3.5 & 0 & 49 & 3.5\\ 6 & 0 & 6 & 6\end{vmatrix}=\dfrac{1}{2}\left(49\times6-6\times3.5\right)=...$		**d**M1
	$\dfrac{273}{2}$ or $136\dfrac{1}{2}$ or 136.5	Any correct exact value	A1
	<u>Special Case:</u> $x=\dfrac{7}{2}\Rightarrow\dfrac{\left(\frac{7}{2}\right)^2}{49}+\dfrac{y^2}{'48'}=1\Rightarrow y=36\Rightarrow$ Area $\Delta OPM=\left(\dfrac{1}{2}\right)\left(\dfrac{7}{'\left(\frac{1}{7}\right)'}-\dfrac{7}{2}\right)(36)=...(819)$ **Scores M0M1A0**		
			(3)
			Total 11

Question Number	Scheme	Notes	Marks
4(a)	$\mathbf{M}\boldsymbol{x}=\lambda\boldsymbol{x}\Rightarrow\begin{pmatrix}0&-1&3\\-1&4&-1\\3&-1&0\end{pmatrix}\begin{pmatrix}1\\-2\\1\end{pmatrix}=\begin{pmatrix}\lambda\\-2\lambda\\\lambda\end{pmatrix}\Rightarrow$ e.g., $2+3=\lambda\Rightarrow\lambda=5$ **or** $(\mathbf{M}-\lambda\mathbf{I})\boldsymbol{x}=0\Rightarrow\begin{pmatrix}-\lambda&-1&3\\-1&4-\lambda&-1\\3&-1&-\lambda\end{pmatrix}\begin{pmatrix}1\\-2\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\Rightarrow$ e.g., $-\lambda+2+3=0\Rightarrow\lambda=5$ M1: Correct method leading to a value for λ A1: Correct value Note that the working may be minimal so e.g. $2+3=\lambda\Rightarrow\lambda=5$ is sufficient. **Correct answer only scores both marks.**		M1 A1
			(2)
(b)	$\begin{pmatrix}0&-1&3\\-1&4&-1\\3&-1&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=-3\begin{pmatrix}x\\y\\z\end{pmatrix}$ **or** $\begin{pmatrix}3&-1&3\\-1&7&-1\\3&-1&3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ **or** e.g., $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}3\\-1\\3\end{pmatrix}\times\begin{pmatrix}-1\\7\\-1\end{pmatrix}$ $\Rightarrow x=\ldots, y=\ldots, z=\ldots$ Uses $\mathbf{M}\boldsymbol{x}=-3\boldsymbol{x}$ or $(\mathbf{M}-(-3)\mathbf{I})\boldsymbol{x}=\mathbf{0}$ to produce simultaneous equations **and** obtains values for x, y and z (not all 0) **or** uses a suitable vector product (with two correct components if method unclear)		M1
	$k\begin{pmatrix}1\\0\\-1\end{pmatrix}$	Any correct eigenvector (allow $x=\ldots$, $y=\ldots$, $z=\ldots$ and apply isw if a vector is subsequently formed incorrectly)	A1
			(2)
(c)	$\mathbf{M}\boldsymbol{x}=\lambda\boldsymbol{x}\Rightarrow$ e.g., $-1(1)+3(1)=\lambda$ $(\mathbf{M}-\lambda\mathbf{I})x=0\Rightarrow$ e.g., $-\lambda-1+3=0$ **or** $\lambda^3-4\lambda^2-11\lambda+30=0$ $\det\mathbf{M}=-30=\lambda_1\lambda_2\lambda_3=-15\lambda$ $\lambda=2$ Correct value. May be seen in their **D** which may come from an attempt at $\mathbf{P}^{\mathrm{T}}\mathbf{MP}$.		B1
	$(\mathbf{D}=)\begin{pmatrix}-3&0&0\\0&'2'&0\\0&0&'5'\end{pmatrix}$	Diagonal matrix with –3 and their eigenvalues anywhere on the leading diagonal and 0's elsewhere. Ignore labelling.	B1ft
	$\begin{pmatrix}1\\-2\\1\end{pmatrix}\rightarrow\begin{pmatrix}\frac{\sqrt6}{6}\\-\frac{\sqrt6}{3}\\\frac{\sqrt6}{6}\end{pmatrix}$ or $'\begin{pmatrix}1\\0\\-1\end{pmatrix}'\rightarrow\begin{pmatrix}\frac{\sqrt2}{2}\\0\\-\frac{\sqrt2}{2}\end{pmatrix}$ or $\begin{pmatrix}1\\1\\1\end{pmatrix}\rightarrow\begin{pmatrix}\frac{\sqrt3}{3}\\\frac{\sqrt3}{3}\\\frac{\sqrt3}{3}\end{pmatrix}$ Correct method seen to normalise **at least one** eigenvector of the two given eigenvectors or their eigenvector from part (b). May be seen in their **P**.		M1
	$\mathbf{D}=\begin{pmatrix}-3&0&0\\0&2&0\\0&0&5\end{pmatrix}$ **and** $\mathbf{P}=\begin{pmatrix}\frac{\sqrt2}{2}&\frac{\sqrt3}{3}&\frac{\sqrt6}{6}\\0&\frac{\sqrt3}{3}&-\frac{\sqrt6}{3}\\-\frac{\sqrt2}{2}&\frac{\sqrt3}{3}&\frac{\sqrt6}{6}\end{pmatrix}$ Both fully correct, consistent and labelled matrices. Elements may not have had denominators rationalised. (Any columns of **P** could be in opposite direction)		A1
			(4)
			Total 8

Note that some candidates go straight into solving $|\mathbf{M}-\lambda\mathbf{I}|=0$ e.g.

$$\begin{vmatrix} -\lambda & -1 & 3 \\ -1 & 4-\lambda & -1 \\ 3 & -1 & -\lambda \end{vmatrix} = 0 \Rightarrow -\lambda\left(\lambda(\lambda-4)-1\right)+3+\lambda+3\left(1-3(4-\lambda)\right)=0$$

$$\Rightarrow \lambda^3-4\lambda^2-11\lambda+30=0 \Rightarrow \lambda=-3,\ 5,\ 2$$

If this is all they do then the B mark in (c) can be awarded for $\lambda = 2$

The other marks in the question are available for the appropriate work.

Question Number	Scheme	Notes	Marks
5(a) **Way 1** **From LHS**	$\left(1-\text{sech}^2 x=\right)1-\left(\dfrac{2}{e^x+e^{-x}}\right)^2$	Replaces sech x with correct expression in terms of exponentials	B1
	$=\dfrac{\left(e^x+e^{-x}\right)^2-4}{\left(e^x+e^{-x}\right)^2}=\dfrac{e^{2x}+2+e^{-2x}-4}{\left(e^x+e^{-x}\right)^2}$	Expresses as a single fraction (or 2 fractions with the same denominator) and expands numerator	M1
	$=\dfrac{\left(e^x-e^{-x}\right)^2}{\left(e^x+e^{-x}\right)^2}=\tanh^2 x$	Fully correct proof	A1*
Way 2 **Diff. of 2 squares**	$1-\text{sech}^2 x=(1+\text{sech}\, x)(1-\text{sech}\, x)=\left(1+\dfrac{2}{e^x+e^{-x}}\right)\left(1-\dfrac{2}{e^x+e^{-x}}\right)$ Uses difference of two squares and replaces sech x with correct expression in terms of exponentials		B1
	$=\left(\dfrac{e^x+e^{-x}+2}{e^x+e^{-x}}\right)\left(\dfrac{e^x+e^{-x}-2}{e^x+e^{-x}}\right)=\dfrac{e^{2x}+1-2e^x+1+e^{-2x}-2e^{-x}+2e^x+2e^{-x}-4}{\left(e^x+e^{-x}\right)^2}$ Expresses as a single fraction and expands numerator		M1
	$=\dfrac{e^{2x}-2+e^{-2x}}{\left(e^x+e^{-x}\right)^2}=\dfrac{\left(e^x-e^{-x}\right)^2}{\left(e^x+e^{-x}\right)^2}=\tanh^2 x$	Fully correct proof	A1*
Way 3 **From RHS**	$\left(\tanh^2 x=\right)\dfrac{\left(e^x-e^{-x}\right)^2}{\left(e^x+e^{-x}\right)^2}$	Replaces tanh x with correct expression in terms of exponentials	B1
	$=\dfrac{e^{2x}-2+e^{-2x}}{\left(e^x+e^{-x}\right)^2}=\dfrac{e^{2x}+2+e^{-2x}}{\left(e^x+e^{-x}\right)^2}-\dfrac{4}{\left(e^x+e^{-x}\right)^2}$ Expands numerator and splits into two fractions		M1
	$=\dfrac{\left(e^x+e^{-x}\right)^2}{\left(e^x+e^{-x}\right)^2}-\left(\dfrac{2}{e^x+e^{-x}}\right)^2=1-\text{sech}^2 x$	Fully correct proof	A1*
			(3)

<u>Allow "meet in the middle" approaches as long as a conclusion is given e.g. lhs = rhs</u>

Example:

$$rhs=\tanh^2 x=\frac{\left(e^{2x}-1\right)^2}{\left(e^{2x}+1\right)^2}\quad\text{or}\quad lhs=1-\text{sech}^2 x=1-\left(\frac{2}{e^x+e^{-x}}\right)^2$$

B1: Replaces tanh x or sech x with a correct expression in terms of exponentials

$$\frac{\left(e^{2x}-1\right)^2}{\left(e^{2x}+1\right)^2}=\frac{e^{4x}-2e^{2x}+1}{e^{4x}+2e^{2x}+1}\quad\text{and}\quad 1-\left(\frac{2}{e^x+e^{-x}}\right)^2=\frac{\left(e^x+e^{-x}\right)^2-4}{\left(e^x+e^{-x}\right)^2}=\frac{e^{2x}-2+e^{-2x}}{e^{2x}+2+e^{-2x}}$$

M1: Makes progress by e.g. removing brackets on *rhs* and expressing *lhs* as a single fraction and expands numerator

$$\frac{e^{2x}-2+e^{-2x}}{e^{2x}+2+e^{-2x}}=\frac{e^{4x}-2e^{2x}+1}{e^{4x}+2e^{2x}+1}\Rightarrow 1-\text{sech}^2 x=\tanh^2 x$$

A1: Correct proof and (minimal) conclusion e.g. "= rhs" etc.

$$1-\operatorname{sech}^2 x = 1-\left(\frac{2}{e^x+e^{-x}}\right)^2 = \frac{e^{2x}+2+e^{-2x}-4}{\left(e^x+e^{-x}\right)^2} = \frac{e^{2x}+e^{-2x}-2}{e^{2x}+e^{-2x}+2} = \frac{\sinh^2 x}{\cosh^2 x} = \tanh^2 x$$

$$1-\operatorname{sech}^2 x = 1-\left(\frac{2}{e^x+e^{-x}}\right)^2 = \frac{e^{2x}+2+e^{-2x}-4}{\left(e^x+e^{-x}\right)^2} = \frac{e^{2x}+e^{-2x}-2}{e^{2x}+e^{-2x}+2} = \tanh^2 x$$

Both score B1M1A0 as we would need to see numerators and denominators factorised.

Note that we will allow an equivalent identity to be proved by exponentials and the given identity deduced e.g.

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x+e^{-x}}{2}\right)^2 - \left(\frac{e^x-e^{-x}}{2}\right)^2$$

B1: Correct exponential form seen for cosh or sinh used

$$= \frac{e^{2x}}{4}+\frac{1}{2}+\frac{e^{-2x}}{4}-\frac{e^{2x}}{4}+\frac{1}{2}-\frac{e^{-2x}}{4} = 1$$

M1: Expands and collects terms

$$\Rightarrow \cosh^2 x - \sinh^2 x = 1 \Rightarrow 1-\operatorname{sech}^2 x = \tanh^2 x$$

A1*: Fully correct work leading to the correct identity

(b)	$\int \tanh^n 3x\,dx = \int \tanh^{n-2} 3x \tanh^2 3x\,dx$ $= \int \tanh^{n-2} 3x\left(1-\text{sech}^2 3x\right)dx$	Splits $\tanh^n 3x$ into $\tanh^{n-2} 3x \tanh^2 3x$ and applies $\tanh^2 3x = 1-\text{sech}^2 3x$	M1
	Do **not** condone $\int \tanh^n 3x\,dx = \int \tanh^{n-2} 3x \tanh^2 3x\,dx = \int \tanh^{n-2} 3x\left(1-\text{sech}^2 x\right)dx$ unless it is clear that $3x$ was intended and is therefore recovered in subsequent work.		
	$= \int \tanh^{n-2} 3x\,dx - \int \tanh^{n-2} 3x\,\text{sech}^2 3x\,dx$ $\int \tanh^{n-2} 3x\,\text{sech}^2 3x\,dx = \frac{1}{3(n-1)}\tanh^{n-1} 3x$ Expands and integrates $\tanh^{n-2} 3x\,\text{sech}^2 3x$ to obtain $\alpha \tanh^{n-1} 3x$ **Or it is possible to use parts for** $\int \tanh^{n-2} 3x\,\text{sech}^2 3x\,dx$ **:** $\int \tanh^{n-2} 3x\,\text{sech}^2 3x\,dx = \frac{1}{3}\tanh 3x \tanh^{n-2} 3x - \frac{1}{3}\int 3\left(n-2\right)\tanh 3x \tanh^{n-3} 3x\,\text{sech}^2 3x\,dx$ $= \frac{1}{3}\tanh^{n-1} 3x - \left(n-2\right)\int \tanh^{n-2} 3x\,\text{sech}^2 3x\,dx$ $\Rightarrow \int \tanh^{n-2} 3x\,\text{sech}^2 3x\,dx = \frac{1}{3\left(n-1\right)}\tanh^{n-1} 3x$ To score it must be a complete method leading to $\alpha \tanh^{n-1} 3x$ as above		**d**M1
	$I_n = I_{n-2} - \frac{1}{3(n-1)}\left[\tanh^{n-1} 3x\right]_0^{\frac{1}{3}\ln 2} = I_{n-2} - \frac{1}{3(n-1)}\left(\frac{e^{2\ln 2}-1}{e^{2\ln 2}+1}\right)^{n-1}$ Introduces I_{n-2} and applies $x = \frac{1}{3}\ln 2$ using a correct exponential definition of tanh or accept use of a calculator if work is correct e.g. $\tanh\left(\ln 2\right) = \frac{3}{5}$		**dd**M1
	$I_n = I_{n-2} - \frac{\left(\frac{3}{5}\right)^{n-1}}{3(n-1)}$ but condone $I_n = I_{n-2} - \frac{\frac{3}{5}^{n-1}}{3(n-1)}$ Fully correct proof. Allow recovery from slips e.g. tanh →tan→tanh or e.g. $3x$ becoming x and then reverting to $3x$ again If there are clear errors that are not recovered score A0.		A1
			(4)

(c)	$I_5 = I_3 - \dfrac{\left(\frac{3}{5}\right)^{5-1}}{3(5-1)} = I_1 - \dfrac{\left(\frac{3}{5}\right)^{3-1}}{3(3-1)} - \dfrac{\left(\frac{3}{5}\right)^{5-1}}{3(5-1)}$ Uses their reduction formula to obtain I_5 in terms of I_1 Note that there may have already been an attempt at I_1 Condone the use of the letter p for the $\frac{3}{5}$ and allow a "made up" p for this mark. This may be implied by e.g. $I_5 = I_3 - \dfrac{\left(\frac{3}{5}\right)^{5-1}}{3(5-1)},\ I_3 = I_1 - \dfrac{\left(\frac{3}{5}\right)^{3-1}}{3(3-1)}$		M1
	$\int \tanh 3x \,\mathrm{d}x = \dfrac{1}{3}\ln(\cosh 3x)$	Integrates to obtain $q\ln(\cosh rx)$ oe e.g. $q\ln(\text{sech } rx)$ Condone q and/or $r = 1$	M1
	$I_5 = \dfrac{1}{3}\ln\left(\dfrac{e^{\ln 2} + e^{-\ln 2}}{2}\right) - \dfrac{\left(\frac{9}{25}\right)}{6} - \dfrac{\left(\frac{81}{625}\right)}{12}$ Applies $x = \dfrac{1}{3}\ln 2$ using correct exponential definition of cosh or uses a calculator if work is correct e.g. $\cosh(\ln 2) = \dfrac{5}{4}$ to obtain a numerical expression for I_5 Must not be in terms of p now and must be using a value of p obtained in part (b)		**dd**M1
	$\dfrac{1}{3}\ln\dfrac{5}{4} - \dfrac{177}{2500}$	Correct answer in correct form (allow a =..., b =..., c =...) Allow –0.0708 for c	A1
			(4)
			Total 11

Note that part (c) is "Hence" so they need to be using the given reduction formula, however, it is possible to find I_3 directly e.g. :

$$I_5 = I_3 - \frac{\left(\frac{3}{5}\right)^{5-1}}{3(5-1)}$$

$$\int \tanh^3 3x \,\mathrm{d}x = \int \left(\tanh 3x - \tanh 3x \,\text{sech}^2 3x\right)\mathrm{d}x = \left[\frac{1}{3}\ln(\cosh 3x) + \frac{1}{6}\text{sech}^2 3x\right]$$

Score **M1** for using the reduction formula to obtain I_5 in terms of I_3 (allow the letter p for the $\frac{3}{5}$ and allow a "made up" p for this mark) **and** then integrating $\tanh^3 3x$ to the correct form e.g. $\alpha \ln(\cosh 3x) + \beta\,\text{sech}^2 3x$ (oe)

The second **M** mark would also score at this point as in the main scheme for integrating $\tanh 3x$ to obtain $q\ln(\cosh rx)$ oe e.g. $q\ln(\text{sech } rx)$

$$\left[\frac{1}{3}\ln(\cosh 3x) + \frac{1}{6}\text{sech}^2 3x\right]_0^{\frac{1}{3}\ln 2} - \frac{\left(\frac{3}{5}\right)^{5-1}}{3(5-1)} = \frac{1}{3}\ln\frac{5}{4} + \frac{1}{6}\times\frac{16}{25} - \frac{1}{6} - \frac{27}{2500}$$

ddM1 for a complete method **using both limits** to obtain a numerical expression for I_5 using the correct exponential definitions or via a calculator.

A1: $\dfrac{1}{3}\ln\dfrac{5}{4} - \dfrac{177}{2500}$

Correct answer in correct form

(allow a =..., b =..., c =...) Allow –0.0708 for c

Question Number	Scheme	Notes	Marks
6(a)	$\pm\overrightarrow{AB}=\pm\left(\begin{pmatrix}-1\\1\\3\end{pmatrix}-\begin{pmatrix}3\\2\\2\end{pmatrix}\right)=\pm\begin{pmatrix}-4\\-1\\1\end{pmatrix},\pm\overrightarrow{AC}=\pm\left(\begin{pmatrix}-2\\4\\2\end{pmatrix}-\begin{pmatrix}3\\2\\2\end{pmatrix}\right)=\pm\begin{pmatrix}-5\\2\\0\end{pmatrix},\pm\overrightarrow{BC}=\pm\left(\begin{pmatrix}-2\\4\\2\end{pmatrix}-\begin{pmatrix}-1\\1\\3\end{pmatrix}\right)=\pm\begin{pmatrix}-1\\3\\-1\end{pmatrix}$ Correct method to obtain two relevant vectors using **subtraction**. You can ignore labelling e.g. if they find $\overrightarrow{BA}$ but call it $\overrightarrow{AB}$		M1
	e.g., $\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}-4\\-1\\1\end{pmatrix}\times\begin{pmatrix}-5\\2\\0\end{pmatrix}=\begin{pmatrix}-2\\-5\\-13\end{pmatrix}$ Correct method to find the vector product of two relevant vectors (if a correct method is not shown, two correct components for their vectors must be obtained)		**d**M1
	e.g., $\begin{pmatrix}3\\2\\2\end{pmatrix}.\begin{pmatrix}2\\5\\13\end{pmatrix}=6+10+26=42$ Attempts the scalar product between their normal vector and any of the position vectors of *A*, *B* or *C*.		**dd**M1
	$2x + 5y + 13z = 42$ oe e.g. $-2x-5y-13z+42=0$	Any correct **Cartesian** equation.	A1
			(4)
(a) alt 1	$\pm\overrightarrow{AB}=\pm\left(\begin{pmatrix}-1\\1\\3\end{pmatrix}-\begin{pmatrix}3\\2\\2\end{pmatrix}\right)=\pm\begin{pmatrix}-4\\-1\\1\end{pmatrix},\pm\overrightarrow{AC}=\pm\left(\begin{pmatrix}-2\\4\\2\end{pmatrix}-\begin{pmatrix}3\\2\\2\end{pmatrix}\right)=\pm\begin{pmatrix}-5\\2\\0\end{pmatrix},\pm\overrightarrow{BC}=\pm\left(\begin{pmatrix}-2\\4\\2\end{pmatrix}-\begin{pmatrix}-1\\1\\3\end{pmatrix}\right)=\pm\begin{pmatrix}-1\\3\\-1\end{pmatrix}$ Correct method to obtain two relevant vectors using subtraction.		M1
	e.g., $\mathbf{r}=\begin{pmatrix}3\\2\\2\end{pmatrix}+\lambda\begin{pmatrix}-4\\-1\\1\end{pmatrix}+\mu\begin{pmatrix}-5\\2\\0\end{pmatrix}\Rightarrow \begin{matrix}x=3-4\lambda-5\mu\\ y=2-\lambda+2\mu\\ z=2+\lambda\end{matrix}\Rightarrow$ e.g. $\lambda=z-2$ Attempts the parametric equation of the plane and uses components to eliminate at least one of their parameters.		**d**M1
	e.g., $\begin{matrix}x=3-4\lambda-5\mu\\ y=2-\lambda+2\mu\\ z=2+\lambda\end{matrix}\Rightarrow$ e.g. $\lambda=z-2\Rightarrow$ e.g. $\mu=\frac{1}{2}(y-4+z)$ Eliminates both of their parameters.		**dd**M1
	e.g. $x=3-4(z-2)-\frac{5}{2}(y-4+z)$	Any correct **Cartesian** equation.	A1
(a) alt 2	$ax+by+cz=1\rightarrow \begin{matrix}3a+2b+2c=1\\ -a+b+3c=1\\ -2a+4b+2c=1\end{matrix}\Rightarrow a=\frac{1}{21},\ b=\frac{5}{42},\ c=\frac{13}{42}$ $\Rightarrow\frac{1}{21}x+\frac{5}{42}y+\frac{13}{42}z=1$ M1: Substitutes the given points to give 3 equations in 3 unknowns dM1: Solves simultaneously to find values for "*a*", "*b*" and "*c*" ddM1: Substitutes back in to obtain a Cartesian equation A1: Any correct equation		

<table>
<tr><td rowspan="6">(b)</td><td>Line DE : $(\mathbf{r} =) \begin{pmatrix}-1\\1\\-2\end{pmatrix} \pm \lambda \begin{pmatrix}2\\5\\13\end{pmatrix}$</td><td>Obtains parametric form for line DE with their normal (or recalculated normal) seen or implied. Allow one slip only.</td><td>M1</td></tr>
<tr><td colspan="2">$14(2\lambda-1)-(5\lambda+1)-17(13\lambda-2)=-66 \Rightarrow \lambda = ...$
Substitutes their parametric form into the equation of Π_2 and solves for λ – can follow M0 provided their parametric form was an attempt at $\overrightarrow{OD} \pm \lambda(\text{their } \mathbf{n})$</td><td>M1</td></tr>
<tr><td>$\lambda = \frac{85}{198}$</td><td>A correct exact value for λ depending on their method e.g. use of
$\mathbf{n} = -2\mathbf{i} - 5\mathbf{j} - 13\mathbf{k}$ gives $\lambda = -\frac{85}{198}$</td><td>A1</td></tr>
<tr><td colspan="2">$DE = \sqrt{(2\times\frac{85}{198})^2 + (5\times\frac{85}{198})^2 + (13\times\frac{85}{198})^2}$
or e.g.
$E = \left(-\frac{14}{99}, \frac{623}{198}, \frac{709}{198}\right) \Rightarrow DE = \sqrt{\left(-1+\frac{14}{99}\right)^2 + \left(1-\frac{623}{198}\right)^2 + \left(-2-\frac{709}{198}\right)^2}$
Correct method to find a numerical expression for distance DE
Requires previous method mark
Note $DE = -\frac{85}{198}\sqrt{(2)^2+(5)^2+(13)^2} = ...$ is ok for this mark</td><td>dM1</td></tr>
<tr><td>$DE = \frac{85\sqrt{22}}{66}$</td><td>Correct exact answer in the required form
or $p = \frac{85}{66}$ or $1\frac{19}{66}$
Not $DE = -\frac{85\sqrt{22}}{66}$</td><td>A1</td></tr>
<tr><td colspan="2"></td><td>(5)</td></tr>
</table>

<u>Beware – Special Case!</u>

An incorrect sign of λ may fortuitously give the correct length DE.

E.g. $\begin{pmatrix}-1\\1\\-2\end{pmatrix} + \lambda \begin{pmatrix}2\\5\\13\end{pmatrix}$ leading incorrectly to $\lambda = -\frac{85}{198}$ would lead in both **d**M1 cases above to $DE = \frac{85\sqrt{22}}{66}$

E.g. $\begin{pmatrix}-1\\1\\-2\end{pmatrix} + \lambda \begin{pmatrix}-2\\-5\\-13\end{pmatrix}$ leading incorrectly to $\lambda = \frac{85}{198}$ would lead in both **d**M1 cases above to $DE = \frac{85\sqrt{22}}{66}$

In such cases score as M1M1A0M1A1ft i.e. we will only penalise it once.

Way 2 **Sim. eqns** **For first three marks**	$(\pm)\left(\frac{x+1}{2}=\frac{y-1}{5}=\frac{z+2}{13}\right)$ $\Rightarrow y=\frac{5}{2}x+\frac{7}{2},\ z=\frac{13}{2}x+\frac{9}{2}$	Obtains Cartesian form for line *DE* with their normal (or recalculated normal) allowing one slip only **and** attempts to find two variables in terms of the other variable	M1
	$14x-\left(\frac{5}{2}x+\frac{7}{2}\right)-17\left(\frac{13}{2}x+\frac{9}{2}\right)=-66$ $\Rightarrow\ x=-\frac{14}{99},\ y=\frac{623}{198},\ z=\frac{709}{198}$	M1: Substitutes into the plane equation and finds $x=...,\ y=...,\ z=...$ A1: Correct exact values $\Rightarrow$ **Way 1** for last two marks	M1 A1
(c)	e.g. $\overrightarrow{AF}.\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}1\\1\\q-2\end{pmatrix}.\begin{pmatrix}2\\5\\13\end{pmatrix}=2+5+13q-26$ or e.g. $\begin{vmatrix}-4&-1&1\\-5&2&0\\1&1&q-2\end{vmatrix}=-4(2(q-2))-5(q-2)-5-2$ or e.g. rule of Sarrus: $\begin{vmatrix}-4&-1&1&-4&-1\\-5&2&0&-5&2\\1&1&q-2&1&1\end{vmatrix}=-4(2(q-2))-5-5(q-2)-2$ Correct method for vector between *F* and *A*, *B* or *C* and finds scalar product with their normal or attempts the scalar triple product to obtain a linear expression in *q*. For the scalar triple product look for at least 2 correct "elements".		M1
	$\frac{1}{6}(13q-19)=\pm12\Rightarrow q=...$ Sets $\frac{1}{6}$ of their expression in *q* equal to one or both of ±12 (or equivalent work e.g. their expression in *q* equal to one or both of ±72) and proceeds to a value for *q*		**d**M1
	$q=7,-\frac{53}{13}$	Correct values. Allow exact equivalents for $-\frac{53}{13}$ e.g. $-4\frac{1}{13}$	A1
			(3)
			Total 12

Question Number	Scheme/Notes			Marks
7(a)	$y = \arccos(\text{sech}\, x)$			
	e.g.:	$\cos y = \text{sech}\, x \Rightarrow$		
	$\frac{dy}{dx} = -\frac{(-\text{sech}\, x \tanh x)}{\sqrt{1-\text{sech}^2 x}}$	$-\sin y \frac{dy}{dx} = -\text{sech}\, x \tanh x$ or, e.g., $-\sin y = -\text{sech}\, x \tanh x \frac{dx}{dy}$	$\cos y = (\cosh x)^{-1} \Rightarrow$ $-\sin y \frac{dy}{dx} = -(\cosh x)^{-2} \sinh x$	M1
	Differentiates to obtain an equation in $\frac{dy}{dx}$ or $\frac{dx}{dy}$ of the correct form e.g. condone coefficient sign errors only.			
	$\frac{dy}{dx} = \frac{\text{sech}\, x \tanh x}{\tanh x}$	$\sqrt{1-\text{sech}^2 x} \frac{dy}{dx} = \text{sech}\, x \tanh x$ $\Rightarrow \tanh x \frac{dy}{dx} = \text{sech}\, x \tanh x$	$\sqrt{1-\text{sech}^2 x} \frac{dy}{dx} = \frac{\sinh x}{\cosh^2 x}$ $\Rightarrow \tanh x \frac{dy}{dx} = \frac{\sinh x}{\cosh^2 x}$	**d**M1
	Uses correct identities to obtain an equation in $\frac{dy}{dx}$ or $\frac{dx}{dy}$ in terms of x only with no roots but accept $\sqrt{\tanh^2 x}$ as "no roots"			
	$\Rightarrow \frac{dy}{dx} = \text{sech}\, x$	$\Rightarrow \frac{dy}{dx} = \text{sech}\, x$	$\frac{dy}{dx} = \frac{\cosh x}{\sinh x} \cdot \frac{\sinh x}{\cosh^2 x}$ $\Rightarrow \frac{dy}{dx} = \text{sech}\, x$	A1*
	Fully correct proof. An equation in $\frac{dy}{dx}$ or $\frac{dx}{dy}$ and exactly two different hyperbolic functions with no roots must be seen before the given answer but accept $\sqrt{\tanh^2 x}$ as "no roots" Withhold this mark for any mathematical error e.g., **clear** use of $\frac{d}{dx}(\arccos x) = +\frac{1}{\sqrt{1-x^2}}$ and $\frac{d}{dx}(\text{sech}\, x) = +\text{sech}\, x \tanh x$ or e.g. hyperbolic functions written as trig functions or vice versa. Allow slips if they are recovered but clear and consistent errors score A0			
	Note: There may be other methods seen, e.g., using exponentials and "meeting in the middle"			
				(3)

(b)	e.g. $\frac{\mathrm{d}}{\mathrm{d}x}(\coth x) = -\operatorname{cosech}^2 x$ or $\frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x}$ or $\frac{-\operatorname{sech}^2 x}{\tanh^2 x}$ or $1-\coth^2 x$ etc. or e.g. $\frac{(\mathrm{e}^x - \mathrm{e}^{-x})^2 - (\mathrm{e}^x + \mathrm{e}^{-x})^2}{(\mathrm{e}^x - \mathrm{e}^{-x})^2}$ or $\frac{2\mathrm{e}^{2x}(\mathrm{e}^{2x}-1) - 2\mathrm{e}^{2x}(\mathrm{e}^{2x}+1)}{(\mathrm{e}^{2x}-1)^2}$ or $\frac{-4}{(\mathrm{e}^x - \mathrm{e}^{-x})^2}$ etc. Correct derivative of coth x in any form. Allow recovery if they write e.g. $-\operatorname{cosec}^2 x$ when $-\operatorname{cosech}^2 x$ is clearly implied by subsequent work.	B1
	e.g., sech x – cosech²x = 0 $\Rightarrow$ sech x = cosech²x $\Rightarrow \frac{1}{\cosh x} = \frac{1}{\sinh^2 x} \Rightarrow$ $a\cosh^2 x + b\cosh x + c = 0$ or $a\operatorname{sech}^2 x + b\operatorname{sech} x + c = 0$ **or** $\operatorname{sech} x - \operatorname{cosech}^2 x = 0 \Rightarrow\Rightarrow \frac{2}{\mathrm{e}^x + \mathrm{e}^{-x}} - \left(\frac{2}{\mathrm{e}^x - \mathrm{e}^{-x}}\right)^2 = 0 \Rightarrow$ $\Rightarrow A\mathrm{e}^{4x} + B\mathrm{e}^{3x} + C\mathrm{e}^{2x} + D\mathrm{e}^x + E = 0$ Sets $\mathrm{f}'(x) = 0$ **and** uses correct identities to obtain a 3TQ in cosh x or sech x **or** substitutes the correct exponential forms and obtains a 5 term quartic in e^x	M1
	$\cosh^2 x - \cosh x - 1 = 0$ or $\operatorname{sech}^2 x + \operatorname{sech} x - 1 = 0$ oe **or** $\Rightarrow \mathrm{e}^{4x} - 2\mathrm{e}^{3x} - 2\mathrm{e}^{2x} - 2\mathrm{e}^x + 1 = 0$ oe Correct quadratic equation or correct quartic equation.	A1
	$\cosh x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} \left(= \frac{1+\sqrt{5}}{2}\right)$ or e.g., $\left(\operatorname{sech} x + \frac{1}{2}\right)^2 - \frac{1}{4} - 1 = 0 \Rightarrow \operatorname{sech} x = \left(\frac{-1+\sqrt{5}}{2}\right)$ Solves quadratic resulting from sech x + their derivative of coth x = 0 Must obtain a **real and exact** value > 1 (or between 0 and 1 if sech used). Apply usual rules. (No need to reject invalid values) If no solving method seen one solution must be consistent with their equation. For the 5 term quartic in e^x progress is unlikely unless they proceed via e.g. $\left(\mathrm{e}^{2x} - (1+\sqrt{5})\mathrm{e}^x + 1\right)^2 = 0$	**d**M1
	$x = \operatorname{arcosh}\left(\frac{1+\sqrt{5}}{2}\right) = \ln\left(\frac{1+\sqrt{5}}{2} + \sqrt{\left(\frac{1+\sqrt{5}}{2}\right)^2 - 1}\right)$ **or** $\frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2} = \frac{1+\sqrt{5}}{2} \Rightarrow \mathrm{e}^{2x} - (1+\sqrt{5})\mathrm{e}^x + 1 = 0 \Rightarrow \mathrm{e}^x = \frac{1+\sqrt{5}+\sqrt{(1+\sqrt{5})^2 - 4}}{2} \Rightarrow x = ...$ Uses correct logarithmic form or exponentials to find x as a ln of an exact value. Exponential definition must be correct and quadratic solving subject to usual rules or consistent with their equation leading to a value of e^x > 0	**dd**M1
	$\Rightarrow x = \ln\left(\frac{1}{2}(1+\sqrt{5}) + \sqrt{\frac{1}{2}(1+\sqrt{5})}\right)$ or accept $x = \ln\left(\frac{1+\sqrt{5}}{2} + \sqrt{\frac{1+\sqrt{5}}{2}}\right)$ Note that $x = \ln\frac{1}{2}(1+\sqrt{5}) + \sqrt{\frac{1}{2}(1+\sqrt{5})}$ scores A0	A1
		(6)
		Total 9

Correct work in (b) leading to:

$$\cosh^2 x - \cosh x - 1 = 0 \Rightarrow \cosh x = \frac{1+\sqrt{5}}{2}$$

$$x = \operatorname{arcosh}\left(\frac{1+\sqrt{5}}{2}\right) = \ln\left(\frac{1+\sqrt{5}}{2} + \sqrt{\frac{1+\sqrt{5}}{2}}\right)$$

With no evidence where the $\sqrt{\frac{1+\sqrt{5}}{2}}$ comes from, scores: B1M1A1dM1ddM0A0

Question Number	Scheme	Notes	Marks
8(a)	$\dfrac{dx}{dy}=\dfrac{y}{4}$ or $2y\dfrac{dy}{dx}=8$ or $\dfrac{dy}{dx}=\left(\dfrac{1}{2}\right)\left(2\sqrt{2}\right)x^{-\frac{1}{2}}$ or $\left(\dfrac{1}{2}\right)\left(2\sqrt{2}\right)\left(\dfrac{2\sqrt{2}}{y}\right)$ **oe** Any correct equation in $\dfrac{dx}{dy}$ or $\dfrac{dy}{dx}$ in terms of y or x		B1
	$\left(\int\sqrt{1+\left(\dfrac{dx}{dy}\right)^2}\,dy=\right)\int\sqrt{1+\left(\dfrac{y}{4}\right)^2}\,(dy)$ or $\left(\int\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}.\dfrac{dx}{dy}dy=\right)\int\sqrt{1+\left(\dfrac{4}{y}\right)^2}.\dfrac{y}{4}(dy)$ Forms $\int\sqrt{1+\left(\dfrac{dx}{dy}\right)^2}\,(dy)$ or $\int\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}.\dfrac{dx}{dy}(dy)$ correctly with their derivative		M1
	$x=18\Rightarrow y^2=144\Rightarrow\beta=12,\ \alpha=24$ $\Rightarrow$(perimeter of $R=$) $24+2\displaystyle\int_0^{12}\sqrt{1+\frac{y^2}{16}}\,dy$	Correct expression	A1
			(3)

(b)	$y = 4\sinh u \Rightarrow \dfrac{dy}{du} = 4\cosh u$	Correct derivative. Condone $\dfrac{dy}{dx} = 4\cosh u$	B1
	$\int \sqrt{1+\dfrac{y^2}{16}}\,dy = \int \sqrt{1+\dfrac{(4\sinh u)^2}{16}}(4\cosh u)(du)$ $\left(= 4\int \cosh^2 u\, du\right)$	Full substitution, correct for their $\dfrac{dy}{du}$	M1
	$\int \cosh^2 u\, du = \int\left(\dfrac{1}{2}\cosh 2u + \dfrac{1}{2}\right)du = \dfrac{1}{4}\sinh 2u + \dfrac{1}{2}u$ **or** $\int\left(\dfrac{e^u + e^{-u}}{2}\right)^2 du = \int\left(\dfrac{e^{2u}}{4} + \dfrac{1}{2} + \dfrac{e^{-2u}}{4}\right)du = \dfrac{e^{2u}}{8} + \dfrac{1}{2}u - \dfrac{e^{-2u}}{8}$ **d**M1: Uses $\cosh^2 u = \pm\dfrac{1}{2}\cosh 2u \pm \dfrac{1}{2}$ and integrates to obtain $a\sinh 2u + bu$ **or** uses $k\left(e^u + e^{-u}\right)$ for cosh u, expands and integrates to obtain $ae^{2u} + bu + ce^{-2u}$ A1: Correct integration		**d**M1 A1
	Perimeter of R:		**dd**M1
	$= 24 + (2)(4)\left[\dfrac{1}{4}\sinh 2u + \dfrac{1}{2}u\right]_0^{\text{arsinh } 3 = \ln(3+\sqrt{10})}$ $= 24 + 2\left[2\sinh u\sqrt{1+\sinh^2 u} + 2u\right]_0^{\text{arsinh}3 = \ln(3+\sqrt{10})}$ $= 24 + 2\left[(2)(3)\sqrt{1+3^2} + 2\ln\left(3+\sqrt{10}\right)\right]$	$= 24 + (2)(4)\left[\dfrac{e^{2u}}{8} + \dfrac{1}{2}u - \dfrac{e^{-2u}}{8}\right]_0^{\ln(3+\sqrt{10})}$ $= 24 + e^{2\ln(3+\sqrt{10})} - e^{-2\ln(3+\sqrt{10})} + 4\ln\left(3+\sqrt{10}\right)$ $24 + \left(3+\sqrt{10}\right)^2 - \dfrac{1}{\left(3+\sqrt{10}\right)^2} + 4\ln\left(3+\sqrt{10}\right)$	
	Substitutes arsinh3 and/or $\ln\left(3+\sqrt{3^2+1}\right)$ into their expression using correct identities or correctly removes exponentials to obtain a numerical expression in constants and lns only Accept use of calculator here e.g. $\sinh(2\text{arsinh}3) = 6\sqrt{10}$		
	$24 + 12\sqrt{10} + 4\ln\left(3+\sqrt{10}\right)$ or, e.g., $4\left(6 + 3\sqrt{10} + \ln\left(3+\sqrt{10}\right)\right)$	Correct answer – any exact simplified equivalent	A1
	Note: Integration by calculator is likely to access the first two marks only		**(6)**
			Total 9
	TOTAL FOR PAPER: 75 MARKS		

Pearson Education Limited. Registered company number 872828
with its registered office at 80 Strand, London, WC2R 0RL, United Kingdom